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CH# 1

Hilbert Spaces

Inner Product

Let V be a vector space (linear space) over the field of real numbers or complex numbers. An inner product on V is a mapping $\langle\ ,\ \rangle : V\times V \longrightarrow F$ (real or complex) which associates a number $\langle u, v\rangle$ of field with each pair of vectors u & v is called the inner product of u & v if the following conditions are satisfied.

(i) $\langle u, u\rangle \geq 0 \quad \forall u \in V$

and $\langle u, u\rangle = 0 \Leftrightarrow u = 0$

(ii) $\langle \alpha u, v\rangle = \alpha\langle u, v\rangle$ OR $\langle u, \alpha v\rangle = \bar{\alpha}\langle u, v\rangle$

(iii) $\langle u, v\rangle = \overline{\langle v, u\rangle}$ for complex field.

$\langle u, v\rangle = \langle v, u\rangle$ for real field.

(iv) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

The ordered pair $(V, \langle\ \rangle)$ is called an inner product space or dot product space.

Note # (1) The inner product (sp) is also called dot product, scalar product or pre hilbert

(2) A real vector space with an inner product is called a real inner product space.

Consequences of Definition of Inner Product

(a) for all $x, y, z \in V$ and $a, b \in F$

$\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$

(b) For all $x, y \in V$, $a \in F$

$$\langle x, ay \rangle = \bar{a}\langle x, y \rangle$$

Here

$$\langle x, ay \rangle = \overline{\langle ay \; x \rangle}$$
$$= \bar{a}\overline{\langle y \; x \rangle}$$
$$= \bar{a}\langle x \; y \rangle$$

(c) $\langle x, ay+bz \rangle = \bar{a}\langle x\, y \rangle + \bar{b}\langle y\, z \rangle$

(d) For all finite sequences $\{x_i\}$, $\{y_i\}$ in V and $\{a_i\}$, $\{b_i\}$ in F

$$\left\langle \sum_{i=1}^{m} a_j x_j . \sum_{i=1}^{n} b_i y_i \right\rangle = \sum_{j=1}^{m} \sum_{i=1}^{n} a_i \bar{b}_i \langle x_i \; y_i \rangle$$

<u>Theorem</u> # Every inner product space X is also a normed space.

<u>Proof</u> # Let X be an inner-product space. Define a function $\|\cdot\| : X \longrightarrow R$ by

$$\|x\| = \sqrt{\langle x \; x \rangle}$$

We show that $\|\cdot\|$ is actually a norm

N_1: obviously $\|x\| = \sqrt{\langle x \; x \rangle} \geq 0 \quad x \in V$

because $\langle x, x \rangle \geq 0$

N_2: $\|\alpha x\|^2 = \langle \alpha x \;\; \alpha x \rangle = \alpha \bar{\alpha} \langle x, x \rangle$

$$= |\alpha|^2 \langle x, x \rangle$$

$$\|\alpha x\| = |\alpha| \langle x \; x \rangle = |\alpha| \, \|x\|$$

N_3: $\|x+y\|^2 = \langle x+y \;\; x+y \rangle$

$$= \langle x, x \rangle + \langle x, y \rangle + \langle y \; x \rangle + \langle y \; y \rangle$$

$= \|x\|^2 + \langle x\ y \rangle + \overline{\langle x\ y \rangle} + \|y\|^2$

$= \|x\|^2 + 2\ Re \langle x\ y \rangle + \|y\|^2$

$\leq \|x\|^2 + 2\ |\langle x\ y \rangle| + \|y\|^2$

$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$ (Schawrz Inequality)

$\leq (\|x\| + \|y\|)^2$

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So

$\|x+y\| \leq \|x\| + \|y\|$

Hence $(X, \|\cdot\|)$ is a normed space. However the converse is not true in general i.e a normed space may not be an inner product space.

Note # $d(u,v) = \|u-v\| = \sqrt{\langle u-v\ \ u-v \rangle}$

Cauchy Schwarz Inequality

For any x, y in an inner product space V

$|\langle x, y \rangle| \leq \|x\|\,\|y\| \rightarrow (1)$

Proof # Let $x = 0, y = 0$. Then (1) holds trivially. So we can suppose that at least one of x & y is non-zero

Then for any $\alpha \in F$

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$0 \leq \langle x + \alpha y\ \ x + \alpha y \rangle$

$= \langle x\ x \rangle + \alpha \langle y\ x \rangle + \bar{\alpha} \langle x\ y \rangle + \alpha\bar{\alpha} \langle y\ y \rangle$

$= \|x\|^2 + \alpha \langle y\ x \rangle + \bar{\alpha} \langle x\ y \rangle + \alpha\bar{\alpha} \|y\|^2$

$= a + \alpha \bar{b} + \alpha b + \alpha\bar{\alpha} c$

$= a + \alpha \bar{b} + \alpha b + \alpha\bar{\alpha} c$

Suppose that $\alpha \neq 0$ so that $y \neq 0$

let $\alpha = -b/c$

$\cdot\ a - b\bar{b}/c - b\bar{b}/c + b\bar{b}/c \geq 0$

So that

$$ac - |b|^2 \geq 0$$

$$\Rightarrow |b|^2 \leq ac.$$

$$\Rightarrow |\langle x, y\rangle|^2 \leq \|x\|^2 \|y\|^2$$

$$\Rightarrow |\langle x\ y\rangle| \leq \|x\|\ \|y\|$$

If $\alpha = 0$ then $y = 0$ so

$$\langle x\ y\rangle = \langle x\ 0\rangle = \langle x\ 0\cdot z\rangle$$
$$= 0\langle x\ z\rangle$$
$$= 0$$

$$\|x\|\cdot\|y\| = 0$$

Thus in all cases

$$|\langle x, y\rangle| \leq \|x\|\ \|y\|$$

<u>Example</u> # Let R^n be the Eucledian space of all n-tuples of real numbers

Define a function $R^n \times R^n \to R$ by

$$(x\ y) \longrightarrow \langle x\ y\rangle$$

where $\langle x\ y\rangle = \sum_{i=1}^{n} x_i y_i$

Then $(R^n, \langle\cdot\ \cdot\rangle)$ is an inner product space.

<u>Sol</u> (1) $\langle x\ x\rangle = \sum_{i=1}^{n} x_i^2 \geq 0$

and $\langle x\ x\rangle = 0 \Leftrightarrow \sum_{i=1}^{n} x_i^2 = 0$

$$\Leftrightarrow x_i^2 = 0 \quad \forall i$$
$$\Leftrightarrow x_i = 0 \quad \forall i$$
$$\Leftrightarrow x = 0$$

(2) For any $\alpha \in R$ & $x, z \in R^n$

$$\langle \alpha x\ z\rangle = \sum_{i=1}^{n} (\alpha x_i) z_i = \alpha \sum x_i z_i = \alpha\langle x\ z\rangle$$

(3) $\langle x\ z\rangle = \sum_{i=1}^{n} x_i z_i = \sum_{i=1}^{n} z_i x_i = \langle z\ x\rangle$

(4) For $x, y, z \in R^n$

$$\langle x+y, z\rangle = \sum (x_i + y_i) z_i = \sum x_i z_i + \sum y_i z_i = \langle x, z\rangle + \langle y\ z\rangle$$

Hence R^n is an inner product space

Problem # In the cauchy Schwartz Inequality

$$|\langle x \ y \rangle| \leq \|x\| \, \|y\|$$

equality holds iff x & y are linearly independent

Sol # Let $|\langle x, y \rangle| = \|x\| \, \|y\| \rightarrow ①$

We shall prove that x & y are linearly independent

Case-I # If either $x = 0$ or $y = 0$, then ① is true and evidently x & y are linearly independent

Case-II # Suppose that

$$x \neq 0 \qquad y \neq 0$$

and ① is true

$$\Rightarrow \langle x \ y \rangle \neq 0 \quad \& \quad \langle y \ y \rangle \neq 0$$

If $\lambda = \dfrac{\langle x \ y \rangle}{\langle y \ y \rangle}$, then $\lambda \neq 0$

Now $\langle x - \lambda y \quad x - \lambda y \rangle$

$$= \langle x \ x \rangle - \frac{|\langle x \ y \rangle|^2}{\langle y \ y \rangle}$$

$$= \|x\|^2 - \frac{\|x\|^2 \|y\|^2}{\|y\|^2} = 0$$

$$\Rightarrow x - \lambda y = 0 \qquad (\because \langle x, x \rangle = 0$$

$$\Rightarrow x = \lambda y \qquad \Leftrightarrow x = 0$$

$\Rightarrow x$ & y are linearly dependent.

Conversely let x & y are L.D. Then we can write

$$x = \alpha y$$

$$|\langle x . y \rangle| = |\langle \alpha y \ y \rangle| = |\alpha \langle y \ y \rangle|$$

$$= |\alpha| \langle y \ y \rangle = |\alpha| \, \|y\|^2$$

$$= (|\lambda| \, \|y\|) \, \|y\|$$

$$= \|\lambda y\| \, \|y\|$$

$$= \|x\| \, \|y\|$$ i.e equality holds.

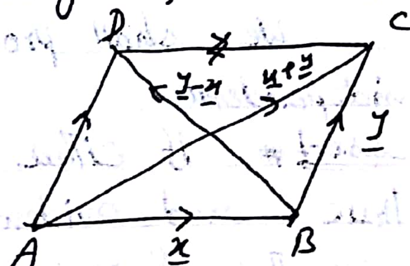
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Parallelogram Law

From elementry geometry we know that the sum of the squares of the lengths of diagonals is equal to the sum of the squares of the lengths of sides.

Thus

$$\overline{AC}^2 + \overline{BD}^2 = 2(\overline{AB}^2 + \overline{BC}^2)$$



In vectorial form, this law assumes the form

$$2(\|x\|^2 + \|y\|^2) = \|x+y\|^2 + \|y-x\|^2$$
$$= \|x+y\|^2 + \|x-y\|^2$$

Theorem # (Parallelogram Law for Inner Product space)

In any inner product space X, for any two elements $u, v \in X$

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

Proof # L.H.S $= \|u+v\|^2 + \|u-v\|^2$

$$= \langle u+v \;\; u+v \rangle + \langle u-v \;\; u-v \rangle$$

$$= \langle u \;\; u \rangle + \langle u \;\; v \rangle + \langle v \;\; u \rangle + \langle v \;\; v \rangle$$
$$+ \langle u \;\; u \rangle + \langle u \;\; -v \rangle + \langle -v \;\; u \rangle + \langle -v \;\; -v \rangle$$

$$= 2\langle u \;\; u \rangle + 2\langle v \;\; v \rangle$$

$$= 2\|u\|^2 + 2\|v\|^2$$

Remarks # The parallelogram law does not hold for any arbitrary normed space. i.e If parallelogram law does not hold, the space cannot be an inner product space.

Example # Space $C[a\ b]$

The space of all continuous functions on $[a\ b]$

Let $X = C[a\ b]$

We know that $C[0\ 2\pi]$ is a Banach space with norm

$$\|f\| = \sup_{0 \le t \le 2\pi} |f(t)|$$

For this norm the parallelogram law i.e

$$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$$

Consider $f(t) = \max(\sin t, 0)$

$$g(t) = \max[-\sin t\ 0]$$

It is clear that

$$\|f\| = \sup_{0 \le t \le 2\pi} |f(t)| = 1$$

$$\|g\| = \sup_{0 \le t \le 2\pi} |g(t)| = 1$$

Also $f(t) + g(t) = \max(\sin t, 0) + \max(-\sin t, 0)$

$$\|f+g\| = \sup_{0 \le t \le 2\pi} |f(t)+g(t)| = 1+0 = 1$$

$$=$$

$$\|f-g\| = \sup_{0 \le t \le 2\pi} |f(t)-g(t)| = 1-0 = 1$$

$$\left\{\begin{array}{lll} f(t) = \max(\sin t, 0) = 1 & & \{0, \ldots\ldots, 1\} \\ g(t) = \max(-\sin t, 0) = 0 & & \{-1, \ldots\ldots 0\} \\ \|g\| = \sup_{0 \le t \le 2\pi} |f(t)| = 1 & & \end{array}\right.$$

$$\|f+g\|^2 + \|f-g\|^2 = 2$$

and $2\|f+g\|^2 + 2\|f-g\|^2 = 4$

$$\|f+g\|^2 + \|f-g\|^2 \neq 2\|f\|^2 + 2\|g\|^2$$

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$\Rightarrow$ Parallelogram Law does not hold in $C[0\ 2\pi]$

$\Rightarrow$ $C[0\ 2\pi]$ is not inner product.

Theorem # A normed space is an inner product space iff it satisfies the $\|$gm law.

Consider again $C[a\ b]$ with the sup-norm

$$\|f\| = \sup_{a \le t \le b} |f(t)|$$

We shall show that $C[a\ b]$ is not an inner product space by showing that $\|$gram Law is not satisfied by its norm.

Take $f(t) = \frac{t-a}{b-a} \longrightarrow ①$

$g(t) = 1 \longrightarrow ②$ from $C[a, b]$

$$\|f\| = \sup_{t \in [a.b]} |f(t)| = 1$$

$$\|g\| = 1$$

$$(g+f)(t) = g(t) + f(t) = 1 + \frac{t-a}{b-a}$$

$$(g-f)(t) = 1 - \frac{t-a}{b-a}$$

$$\|g+f\| = \sup_{t \in [a.b]} \left(1 + \frac{t-a}{b-a}\right) = 1+1 = 2$$

$$\|g-f\| = \sup_{t \in [a.b]} \left(1 - \frac{t-a}{b-a}\right) = 1-0 = 1$$

$$\|g+f\|^2 + \|g-f\|^2 = 5$$

$$2\left[\|g\|^2 + \|f\|^2\right] = 2(1+1) = 4$$

Thus $\|g+f\|^2 + \|g-f\|^2 \neq 2\|g\|^2 + 2\|f\|^2$

$\Rightarrow$ $C[a\ b]$ is not an inner product space.

Corollaries of Parallelogram Law

Corollary # 1: For a real inner product space V

$$\langle x, y\rangle = \frac{1}{4}\{\|x+y\|^2 - \|x-y\|^2\} \quad \forall x, y \in V$$

Proof #

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$
$$= \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2$$
$$= \|x\|^2 + \langle x, y\rangle + \langle x, y\rangle + \|y\|^2$$
$$= \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 \rightarrow (1)$$

$$\|x-y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 \rightarrow (2)$$

(1) − (2) ⇒

$$\|x+y\|^2 - \|x-y\|^2 = 4\langle x, y\rangle$$
$$\langle x, y\rangle = \frac{1}{4}[\|x+y\|^2 - \|x-y\|^2]$$

Corollary # 2 For a Complex inner product space V

(i) $\text{Re}\langle x, y\rangle = \frac{1}{4}[\|x+y\|^2 - \|x-y\|^2]$

(ii) $\text{Im}\langle x, y\rangle = \frac{1}{4}[\|x+iy\|^2 - \|x-iy\|^2]$

Proof #

$$\|x+y\|^2 = \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2$$
$$= \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle}$$

$$\|x+y\|^2 - [\|x\|^2 + \|y\|^2] = 2\,\text{Re}\langle x, y\rangle$$

using parallelogram Law

$$\|x+y\|^2 - \frac{1}{2}[\|x+y\|^2 + \|x-y\|^2] = 2\,\text{Re}\langle x, y\rangle$$

$$\frac{1}{2}[\|x+y\|^2 - \|x-y\|^2] = 2\,\text{Re}\langle x, y\rangle$$

$$\text{Re}\langle x, y\rangle = \frac{1}{4}[\|x+y\|^2 - \|x-y\|^2]$$

(ii) $\|x+iy\|^2 = \|x\|^2 + \langle x, iy\rangle + \langle iy, x\rangle + \|iy\|^2$

$$= \|x\|^2 + \|iy\|^2 + \bar{i}\langle x, y\rangle + i\langle y, x\rangle$$

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$$\Rightarrow \|x+iy\|^2 - (\|x\|^2+\|y\|^2) = -i\langle x\ y\rangle + i\overline{\langle x\ y\rangle}$$

using parallelogram inequality.

$$\|x+iy\|^2 - \frac{1}{2}[\|x+iy\|^2 + \|x-iy\|^2]$$
$$= -i[\langle x\ y\rangle - \overline{\langle x\ y\rangle}]$$

$$\frac{1}{2}[\|x+iy\|^2 - \|x-iy\|^2]$$
$$= -i[2i\,Im\langle x\ y\rangle]$$
$$= 2\,Im\langle x\ y\rangle$$

$$Im\langle x\ y\rangle = \frac{1}{4}\{\|x+iy\|^2 - \|x-iy\|^2\}$$

Corollary# 3 For any x, y, z in an inner product space

$$\|z-x\|^2 + \|z-y\|^2 = \frac{1}{2}\|x-y\|^2 + 2\|z-\frac{1}{2}(x+y)\|^2$$

(Appolonius Identity)

Proof# From ||gram Law

$$\|x'+y'\|^2 + \|x'-y'\|^2 = 2\|x'\|^2 + 2\|y'\|^2$$

Let $x' = z-x \qquad y' = z-y$

$$\|z-x+z-y\|^2 + \|z-x-(z-y)\|^2 = 2\|z-x\|^2 + 2\|z-y\|^2$$

$$\Rightarrow \|2z-(x+y)\|^2 + \|x-y\|^2 = 2\|z-x\|^2 + 2\|z-y\|^2$$

$$\Rightarrow \|z-x\|^2 + \|z-y\|^2 = \frac{1}{2}\|2z-(x+y)\|^2 + \frac{1}{2}\|x-y\|^2$$

$$= \|z-\frac{1}{2}(x+y)\|^2 + \frac{1}{2}\|x-y\|^2$$

Theorem# For any two elements x & y in an inner-product space V

$$\langle x\ y\rangle = \frac{1}{4}\{\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\}$$

(Polarization Identity)

$$\|x+y\|^2 = \langle x+y,\ x+y\rangle$$

$$= \langle x\ x\rangle + \langle x\ y\rangle + \langle y, x\rangle + \langle y\ y\rangle \rightarrow ①$$

$$\|x-y\|^2 = \langle x-y\quad x-y\rangle = \langle x\ x\rangle - \langle x\ y\rangle - \langle y, x\rangle + \langle y, y\rangle$$

$$-\|x-y\|^2 = -\langle x\ x\rangle + \langle x y\rangle + \langle y\ x\rangle - \langle y\ y\rangle \rightarrow ②$$

$$\|x+iy\|^2 = \langle x+iy,\ x+iy\rangle$$

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$$= \langle x\ x\rangle - i\langle x y\rangle + i\langle y\ x\rangle + \langle y\ y\rangle$$

$$i\|x+iy\|^2 = i\langle x\ x\rangle + \langle x\ y\rangle - \langle y\ x\rangle + i\langle y, y\rangle \rightarrow ③$$

$$\|x-iy\|^2 = \langle x-iy\ ,\ x-iy\rangle$$

$$= \langle x, x\rangle + i\langle x\ y\rangle - i\langle y, x\rangle + \langle y, y\rangle$$

$$-i\|x-iy\|^2 = -i\langle x\ x\rangle + \langle x, y\rangle - \langle y, x\rangle - i\langle y, y\rangle \rightarrow ④$$

By Adding ① , ② ③ & ④

$$\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$

$$= 4\langle x\ y\rangle$$

$$\langle x\ y\rangle = \frac{1}{4}\left\{\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right\}$$

Result # (i) $Re\langle x\ y\rangle = \frac{1}{4}\left\{\|x+y\|^2 - \|x+y\|^2\right\}$

(ii) $Im\langle x\ y\rangle = \frac{1}{4}\left\{\|x+iy\|^2 - \|x-iy\|^2\right\}$

Remarks # We have proved already that every inner product space is a normed space. But the converse is not true in general i.e every normed space is not an inner product space. We give a Counter-example

Example # Consider the space $C[0\ \pi/2]$ of real valued functions defined on $[0\ \pi/2]$. Then $C[0\ \pi/2]$ is a normed space under the sup-norm defined by

$$\|f\| = \sup_{x\in[0\ \pi/2]} |f(x)| \longrightarrow (1)$$

Consider the functions $f, g \in C[0\ \pi/2]$ given by

$$f(t) = \sin t \qquad g(t) = \cos t \qquad t\in[0\ \pi/2]$$

Then $f, g \in C[0\ \pi/2]$

Suppose that $C[0\ \pi/2]$ is an inner product and let an inner-product $\langle \cdot\ \cdot \rangle$ define the norm by (1). Then the parallelogram law is satisfied in $C[0\ \pi/2]$. However

$$\|f\| = 1 = \|g\|$$

and

$$\|f+g\| = \sup_{t\in[0\ \pi/2]} |f(t)+g(t)| = \sup_{t\in[0\ \pi/2]} (|\sin t + \cos t|)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

while

$$\|f-g\| = 1$$

Hence

$$\|f+g\|^2 + \|f-g\|^2 = 3 \neq 4 = 2\|f\|^2 + 2\|g\|^2$$

a contradiction.

Hence $C[0\ \pi/2]$ with sup norm is not an inner product space.

Another example of normed space which is not an inner product space is l^p, $p>1$ and $p\neq 2$

Here we can take.

$x = (1, 1, 0, 0 \ldots)$

$y = (1, -1, 0, 0 \ldots)$

So that

$x+y = (2, 0, 0, 0 \ldots)$

$x-y = (0, 2, 0, 0 \ldots)$

$\|x+y\|^2 + \|x-y\|^2 = (2^p)^{1/p} + (2^p)^{1/p} = 4.$

$\|x\| = (|1|^p + |1|^p)^{1/p} = 2^{1/p}$

$\|x\|^2 = 2^{2/p}$

$\|y\| = (|1|^p + |-1|^p)^{1/p} = 2^{1/p}$

$\|y\|^2 = 2^{2/p}$

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$2\|x\|^2 + 2\|y\|^2 = 2 \cdot 2^{2/p} + 2 \cdot 2^{2/p}$

$= 4 \cdot 2^{2/p}$

$\|x+y\|^2 + \|x-y\|^2 \neq 2\|x\|^2 + 2\|y\|^2$

Inner Product Space As Metric Space

We know that every inner product space is a normed space with norm induced by inner product and every normed space is a metric space under a metric induced by its norm. Therefore every inner product space V is a metric space. The metric in V is defined by

$$d(x, y) = \|x-y\| = \sqrt{\langle x-y, x-y \rangle} \quad \forall x, y \in V$$

Remarks # The notions of continuity, Cauchy sequence, convergence of a sequence, limit of sequence, open and closed balls carry over to the inner product spaces. As such we can talk about complete metric spaces.

Hilbert Space

A Complete inner Product space is called a Hilbert space.

The Hilbert Space l^2

The space l^2 of all Complex sequences $x = \{x_i\}$ such that

$$\sum_{i=1}^{\infty} |x_i|^2 < \infty$$

is an inner product space under the inner Product defined by

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i \qquad \rightarrow (1)$$

To see that inner product is well defined, we using Holder's inequality observe that

$$\left|\sum_{i=1}^{n} x_i \bar{y}_i\right| \leq \sum_{i=1}^{n} |x_i \bar{y}_i| \leq \sqrt{\sum_{i=1}^{n} |x_i|^2} \cdot \sqrt{\sum_{i=1}^{n} |y_i|^2}$$

So the Series on R.H.S (1) is absolutely Convergent other properties of inner-Product space can be easily verified.
The Completeness of l^2 was already established.

Convex Set

A subset C of a normed space X is called convex if for all $x, y \in C$
$\alpha \in [0 \;\; 1]$

$$\alpha x + (1-\alpha) y \in C \qquad \alpha + 1 - \alpha = 1$$

OR $$\alpha x + \beta y \in C \qquad \alpha, \beta \in [0 \;\; 1], \alpha + \beta = 1$$

Theorem # A closed convex set of a Hilbert space H contains a unique vector of smallest norm.

Proof # Intuitively, a convex set is a non-empty set which contains the segment joining any pair of its points i.e a convex subset S in a normed space is a set with property that if x, y are in S, then

$$z = x + t(y-x) = (1-t)x + ty \text{ is also in } S$$

for every real number t such that $0 \le t \le 1$

Now since C is a convex, it is non-empty and contains $\frac{1}{2}(x+y)$ whenever it contains x & y. Let

$$d = \operatorname{Inf}\{\|x\| : x \in C\}$$

Then there exists a sequence x_n of vectors in C such that $\|x_n\| \to d$.

By convexity of C,

$$\frac{x_m + x_n}{2} \in C$$

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and $\left\|\frac{x_m + x_n}{2}\right\| \ge d$

$$\Rightarrow \|x_m + x_n\| \ge 2d.$$

Using the parallelogram law

$$\|x_m - x_n\|^2 = 2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m + x_n\|^2$$

$$\le 2\|x_m\|^2 + 2\|x_n\|^2 - 4d^2$$

Since $2\|x_m\|^2 + 2\|x_n\|^2 - 4d^2 \to 2d^2 + 2d^2 - 4d^2 = 0$

It follows that $\{x_n\}$ is a cauchy sequence in C.

Since H is complete and C is closed, C is ~~close~~ complete and there exists a vector x in C such that

$$x_n \to x$$

It is clear by the fact that

$$\|x\| = \|\lim_{n\to\infty} x_n\| = \lim_{n\to\infty}\|x_n\| = d$$

i.e x is a vector in C with the smallest norm.

Uniqueness # To see the uniqueness of x, suppose that x' is a vector in C other than x which also has norm d.

Then $\frac{x+x'}{2} \in C$ ($\because C$ is convex)

By parallelogram

$$\left\|\frac{x+x'}{2}\right\|^2 = \frac{\|x\|^2}{2} + \frac{\|x'\|^2}{2} - \left\|\frac{x-x'}{2}\right\|^2$$

$$< \frac{\|x\|^2}{2} + \frac{\|x'\|^2}{2} = d^2$$

which contradicts the definition of d.

Hence x is unique.

Remarks # Usually we consider the complex inner product space in defining the Hilbert space but there are few places where complex scalars are necessary. However it is only in complex case that theory of operators on a Hilbert space assumes really satisfactory form

Theorem # In any inner product space V

(a)# (Continuity of inner product)

$x_n \to x$, $y_n \to y$ implies $\langle x_n, y_n\rangle \to \langle x, y\rangle$

(b) # If $\{x_n\}$, $\{y_n\}$ are Cauchy sequences in V, then $\langle x_n, y_n\rangle$ is convergent in F (R or C)

Proof # For any natural number n

$$|\langle x_n, y_n\rangle - \langle x, y\rangle| = |\langle x_n - x, y_n - y\rangle + \langle x, y_n - y\rangle + \langle x_n - x, y\rangle| \quad \to ①$$

$\leq |\langle x_n - x, y_n - y\rangle| + |\langle x, y_n - y\rangle| + |\langle x_n - x, y\rangle|$

$\leq \|x_n - x\| \, \|y_n - y\| + \|x\| \, \|y_n - y\| + \|x_n - x\| \, \|y\|$

Thus if $x_n \to x$, $y_n \to y$, Then

$\|x_n - x\| \to 0$ $\|y_n - y\| \to 0$ as $n \to \infty$

So that

$|\langle x_n, y_n\rangle - \langle x, y\rangle| \to 0$ as $n \to \infty$

Hence

$$\langle x_n, y_n\rangle \to \langle x, y\rangle$$

(b) Suppose that $\{x_n\}$, $\{y_n\}$ are cauchy sequences in V. Then for natural numbers m, n

$\|x_n - x_m\| \to 0$. $\|y_n - y_m\| \to 0$ as $m, n \to \infty$

Hence as above.

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$|\langle x_n, y_n\rangle - \langle x_m, y_m\rangle|$

$= |\langle x_n - x_m, y_n - y_m\rangle + \langle x_m, y_n - y_m\rangle + \langle x_n - x_m, y_m\rangle|$

$\leq |\langle x_n - x_m, y_n - y_m\rangle| + |\langle x_m, y_n - y_m\rangle| + |\langle x_n - x_m, y_m\rangle|$

$\leq \|x_n - x_m\| \, \|y_n - y_m\| + \|x_n\| \, \|y_n - y_m\| + \|x_n - x_m\| \, \|y_m\|$ → (1)

Since every cauchy sequence is bounded, R.H.S tends to zero as m, n tend to ∞. Hence $\langle x_n, y_n\rangle$ is a cauchy sequence in F. Since F is R or C, this sequence converges in F

Corollary # In an inner product space V

(i) $x_n \to x \Rightarrow \|x_n\| \to \|x\|$

(ii) If $\{x_n\}$ is a cauchy sequence, then $\|x_n\|$ converges.

Proof (i) It follows by the continuity of the norm

(ii) # If $\{x_n\}$ is cauchy sequence, $\|x_n\|$ is cauchy sequence in F (R or C) and is therefore

is convergent because of the completeness of $F(\mathbb{R}$ or $\mathbb{C})$

Theorem # Show that for a sequence $\{x_n\}$ in an inner product space. The conditions $\|x_n\| \to \|x\|$ & $\langle x_n, x\rangle \to \langle x, x\rangle$ imply convergence $x_n \to x$

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Sol # Consider

$$\|x_n - x\|^2 = \langle x_n - x, x_n - x\rangle$$

$$= \|x_n\|^2 - \langle x_n, x\rangle - \langle x, x_n\rangle + \|x\|^2$$

as $\langle x_n, x\rangle \to \langle x, x\rangle = \|x\|^2$

$\langle x, x_n\rangle = \overline{\langle x_n, x\rangle} \to \overline{\langle x, x\rangle} = \|x\|^2$

& given that

$$\|x_n\| \to \|x\|$$

$\|x_n - x\|^2 \to \|x\|^2 - \|x\|^2 - \|x\|^2 + \|x\|^2$ as $n \to \infty$

$\|x_n - x\|^2 \to 0$ as $n \to \infty$

$\Rightarrow x_n \to x$ as $n \to \infty$

Theorem # Let $x_n \to x$ & $y_n \to y$ in the Hilbert space X and $\alpha_n \to \alpha$ where α_n's and α are scalars. Then show that

(i) $x_n + y_n \to x + y$

(ii) $\alpha_n x_n \to \alpha x$

Proof # $\|(x_n + y_n) - (x + y)\|^2$

$= \langle (x_n + y_n) - (x + y), (x_n + y_n) - (x + y)\rangle$

$= \langle (x_n - x) + (y_n - y), (x_n - x) + (y_n - y)\rangle$

$= \langle (x_n - x), (x_n - x)\rangle + \langle (x_n - x), (y_n - y)\rangle + \langle (y_n - y), (x_n - x)\rangle$
$+ \langle (y_n - y), (y_n - y)\rangle \quad \to (1)$

Since $x_n \to x$ and $y_n \to y$, Therefore

$\|x_n - x\|^2 = \langle x_n - x, x_n - x\rangle \to 0$ as $n \to \infty$

and $\|y_n - y\|^2 = \langle y_n - y, y_n - y\rangle \to 0$ as $n \to \infty$

By Cauchy Schwartz Inequality

~~$\langle x_n, y_n\rangle$~~ $|\langle x_n - x, y_n - y\rangle| \le \|x_n - x\| \|y_n - y\| \to 0$ as $n \to \infty$

and

$|\langle y_n - y, x_n - x\rangle| \le \|y_n - y\| \|x_n - x\| \to 0$ as $n \to \infty$

using these relations in ①, we have.

$$\|(x_n + y_n) - (x + y)\| \to 0 \text{ as } n \to \infty$$

Thus $x_n + y_n \to x + y$

(ii) $\|\alpha_n x_n - \alpha x\|^2 = \langle \alpha_n x_n - \alpha_n, \alpha_n x_n - \alpha_n\rangle$

$= \langle (\alpha_n x_n - \alpha x_n) + (\alpha x_n - \alpha x), (\alpha_n x_n - \alpha x_n) + (\alpha x_n - \alpha x)\rangle$

$= \langle \alpha_n x_n - \alpha x_n, \alpha_n x_n - \alpha x_n\rangle + \langle \alpha_n x_n - \alpha x_n, \alpha x_n - \alpha x\rangle$

$+ \langle \alpha x_n - \alpha x, \alpha_n x_n - \alpha x_n\rangle + \langle \alpha x_n - \alpha x, \alpha x_n - \alpha x\rangle$

$\le \|\alpha_n x_n - \alpha x_n\|^2 + |\alpha|^2 \|x_n\| \|x_n - \alpha\| + \|\alpha x_n - \alpha x\|$

$\|\alpha_n x_n - \alpha x_n\| + \|\alpha x_n - \alpha x\| \|\alpha x_n - \alpha x\|$

$\le \|\alpha_n - \alpha\|^2 \|x_n\|^2 + |\alpha|^2 \|x\| \|\alpha_n - \alpha\|$

$+ \{ |\alpha| \|x_n - x\| |\alpha_n - \alpha| \|x_n\| + |\alpha| \|x_n - x\| |\alpha_n - \alpha| \}$

$\to 0$ as $n \to \infty$

$\Rightarrow \alpha_n x_n \to \alpha x$ as $n \to \infty$

Orthogonality In Inner Product Space (Hilbert space)

Two vectors x & y in an inner product (Hilbert) space are said to be orthogonal or perpendicular to each other if

$$\langle x, y\rangle = 0$$

Remarks # (1) orthogonality is usually denoted by $\perp$ (perp) & we have

$$x \perp y \Leftrightarrow y \perp x$$

(2) $x \perp 0$ for every x

(3) # 0 is only vector orthogonal to itself because for any non-zero vector x

$$\langle x, x \rangle = \|x\|^2 \neq 0$$

Orthogonal Set

A subset (A) of an inner product space (Hilbert space) is said to be an orthogonal set if any two distinct vectors in (A) are orthogonal.

Orthonormal Set

A subset (A) of an inner product space is said to be an orthonormal set if A is orthogonal set for which $\|x\| = 1$ $\forall x \in A$ i.e in which every vector has unit norm.

OR

A is an orthonormal set if

$$\langle x, y \rangle = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases}$$

$$= \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Orthonormal Sets

Two set A & B in an inner product space are said to be orthonormal if

$$\langle u_n, u_m \rangle = \delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

$\{u_n\} \in A$ $\{u_m\} \in B$

A Vector Orthogonal to a Set

A vector x is said to be orthogonal to a non-empty set A in an inner product space (Hilbert spac) and is written as $x \perp A$ if

$$\langle x \;\; y\rangle = 0 \qquad \forall y \in A$$

Orthonormal System

A set $\{x_\alpha : \alpha \in I\}$ of non-zero vectors in an inner product space V is said to be an orthonormal system if

$$\langle x_\alpha , x_\beta \rangle = \delta_{\alpha\beta} \qquad \alpha, \beta \in I$$

If $\{x_\alpha : \alpha \in I\}$ is an orthonormal system, then for any $\alpha, \beta \in I$

$$\| x_\alpha - x_\beta \| = \sqrt{2}$$

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Angle between Two Vectors

Let V be a real inner product space and $x, y \in V$, then angle θ bet x & y is defined by

$$\cos\theta = \frac{\langle x , y\rangle}{\|x\| \, \|y\|} \rightarrow ① \quad (0 \leq \theta \leq \pi)$$

By Cauchy Schwarz inequality

$$|\langle x \;\; y\rangle| \leq \|x\| \, \|y\|$$

$$|\cos\theta| = \frac{|\langle x \;\; y\rangle|}{\|x\| \, \|y\|} \leq \frac{\|x\| \, \|y\|}{\|x\| \, \|y\|} = 1$$

$$\Rightarrow \quad -1 \leq \cos\theta \leq 1$$

Thus ① actually defines an angle between x & y

Remarks# The angle is not defined in Complex inner product space but the concept of orthogonality keeps its meaning.

Examples

1) In R^n the elements

$e_1 = (1, 0, 0 \ldots 0)$

$e_2 = (0, 1, 0, 0 \ldots 0)$

$\vdots$

$e_n = (0, 0, 0 \ldots 1)$

form an orthonormal system/set

Norm in R^n is defined as

$$\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2} \quad \text{and}$$

$$\langle e_i \; e_j \rangle = 0 \quad i \neq j$$
$$= 1 \quad i = j$$

2) Consider the space $C[0 \; 2\pi]$ of Continuous functions defined on $[0 \; 2\pi]$. Define an inner product in $C[0 \; 2\pi]$ by

$$\langle f \cdot g \rangle = \int_0^{2\pi} f(x)\, g(x)\, dx \quad (\text{for real valued functions})$$

$$= \int_0^{2\pi} f(x)\, \overline{g(x)}\, dx \quad (\text{for Complex valued functions})$$

Then the system

$\{ S_n(x),\; C_0(x),\; C_n(x) : n = 1, 2, \ldots \}$ where

$$S_n = \frac{1}{\sqrt{\pi}} \sin nx \quad n = 1, 2 \ldots$$

$$C_0(x) = \frac{1}{\sqrt{2\pi}}$$

$$C_n(x) = \frac{1}{\sqrt{\pi}} \cos nx \quad n = 1, 2 \ldots$$

is an orthonormal system

Here

$$\langle S_n(x) \; S_m(x) \rangle = \frac{1}{\pi} \int_0^{2\pi} \sin nx \sin mx \, dx$$

$$= \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

$$\langle C_0(x) \; S_n(x) \rangle = 0$$

$$\langle C_0(x), C_0(x)\rangle = \frac{1}{2\pi}\int_0^{2\pi} 1\cdot dx = \frac{1}{2\pi}[x]_0^{2\pi} = 1$$

$$\langle C_n(x), C_m(x)\rangle = \delta_{mn} = \begin{cases} 1 & m=n \\ 0 & m\neq n\end{cases}$$

$$\langle C_0(x)\quad C_n(x)\rangle = 0$$

$$\langle C_n(x)\quad C_m(x)\rangle = \frac{1}{\pi^2}\int_0^{2\pi} \cos nx\cos mx\,dx \quad \text{if } m\neq n$$

$$= \frac{1}{\pi^2}\cdot\frac{1}{2}\int_0^{2\pi} 2\cos nx\cos mx\,dx$$

$$= \frac{1}{2\pi^2}\int_0^{2\pi}[\cos(m+n)x + \cos(m-n)x]\,dx$$

$$= \frac{1}{2\pi^2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_0^{2\pi}$$

$$= 0$$

If $m=n$

$$\langle C_n(x), C_m(x)\rangle = \frac{1}{\pi}\int_0^{2\pi} \cos^2 nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{1+\cos 2nx}{2}\right)dx$$

$$= \frac{1}{\pi}\left[\frac{1}{2}x + \frac{\sin 2nx}{4n}\right]_0^{2\pi}$$

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$$= \frac{1}{\pi}\{\pi\} = 1$$

other requirements can be similarly verified.

3)) For any orthogonal system $\{x_1, x_2, \dots x_n\}$ the vectors

$$e_i = \frac{x_i}{\|x_i\|} \qquad i=1,2,\dots,n$$

form an orthonormal system.

Theorem

(a) (Pythagorian Theorem)

If x & y are orthogonal vector in any inner-product space V, then

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

(b) (Generalized Pythagorian rule)

If $\{x_1, x_2 - - - - x_n\}$ is an orthogonal set of vectors in an inner product space V, then

$$\|x_1 + x_2 + - - - x_n\|^2 = \|x_1\|^2 + \|x_2\|^2 + - - + \|x_n\|^2$$

$$\|\sum_{i=1}^{n} x_i\|^2 = \sum_{i=1}^{n} \|x_i\|^2$$

Proof # (a) $\because$ $x \perp y$

$\therefore$ $\langle x, y\rangle = 0$

$$\|x+y\|^2 = \langle x+y \quad x+y\rangle$$

$$= \langle x \quad x\rangle + \langle x \quad y\rangle + \langle y, x\rangle + \langle y \quad y\rangle$$

$$= \|x\|^2 + 0 + 0 + \|y\|^2$$

$$= \|x\|^2 + \|y\|^2$$

(b)

$$\|x_1 + x_2 + x_3 + - - - + x_n\|^2$$

$$= \langle x_1 + x_2 + - - - + x_n ,\ x_1 + x_2 + x_3 + - - - + x_n\rangle$$

$$= \langle \sum_{i=1}^{n} x_i \quad \sum_{j=1}^{n} x_j\rangle$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} \langle x_i \quad x_j\rangle$$

$$= \sum_{i=1}^{n} \langle x_i \quad x_i\rangle \qquad \langle x_i \quad x_j\rangle = 0 \quad i \neq j$$

$$= \sum_{i=1}^{n} \|x_i\|^2$$

Theorem # Any orthogonal set of vectors in an inner product space is linearly independent

OR

Any sequence $\{x_n\}$ of non-zero mutually orthogonal vectors in an inner product space is linearly independent

Proof # It is enough to show that for any natural number n the set

$$\{x_1, x_2 \ldots x_n\} \text{ is L.I}$$

Let

$$\sum_{i=1}^{n} \alpha_i x_i = 0$$

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Multiplying both sides by x_j

$$\left\langle \sum_{i=1}^{n} \alpha_i x_i \;\; x_j \right\rangle = \langle 0 \;\; x_j \rangle$$

$$\sum_{i=1}^{n} \alpha_i \langle x_i \;\; x_j \rangle = 0$$

$$\Rightarrow \alpha_j \langle x_j \;\; x_j \rangle = 0$$

$$\Rightarrow \alpha_j \|x_j\|^2 = 0$$

$$\Rightarrow \alpha_j = 0 \qquad \because \|x_j\|^2 \neq 0$$

$$\Rightarrow \alpha_j = 0 \quad j = 1, 2, \ldots$$

Hence $\{x_1, x_2 \ldots x_n\}$ is L.I

Remarks # Since an orthonormal system is also an orthogonal system, therefore any orthonormal system in an inner product space is also linearly independent

Theorem # If a non-zero vector x in an inner product space is orthogonal to a subset A of the inner product space, then x is orthogonal to any L.C of vectors in A

Proof # Let $\sum_{i=1}^{n} \alpha_i y_i$ be a L.C of vectors y_i in A

Then.

$$\langle x \quad \sum_{i=1}^{n} \alpha_i y_i \rangle = \sum_{i=1}^{n} \bar{\alpha}_i \langle x \quad y_i \rangle$$

$$= \sum_{i=1}^{n} \bar{\alpha}_i (0)$$

$$= 0$$

Hence $x \perp \sum_{i=1}^{n} \alpha_i y_i$.

Thus x is orthogonal to any linear Combination of vectors in A.

Exercise # If $x_1, x_2, \ldots x_n$ are mutually orthogonal vectors in an inner product space, then

$$\|\sum_{i=1}^{n} \alpha_i x_i\|^2 = \sum_{i=1}^{n} |\alpha_i|^2 \|x_i\|^2.$$

Sol # $L.H.S = \|\sum_{i=1}^{n} \alpha_i x_i\|^2$

$$= \langle \sum_{i=1}^{n} \alpha_i x_i \quad \sum_{j=1}^{n} \alpha_j x_j \rangle$$

$$= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{n} \bar{\alpha}_j \langle x_i \quad x_j \rangle$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\alpha}_j \langle x_i \quad x_j \rangle$$

$$= \sum_{i=1}^{n} |\alpha_i|^2 \, \|x_i\|^2$$

$= R.H.S$

Theorem# If $\{v_1, v_2, \dots, v_n\}$ is an orthonormal basis for an inner product space V, and U is any vector in V, then

$$U = \langle U, v_1\rangle v_1 + \langle U, v_2\rangle v_2 + \dots + \langle U, v_n\rangle v_n$$

Proof# Since $\{v_1, v_2, \dots, v_n\}$ is a basis U can expressed in the form

$$U = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \rightarrow ①$$

We show that for α_i we have

$$\alpha_i = \langle U, v_i\rangle \quad i = 1, 2, \dots, n$$

from ①

$\langle U, v_i\rangle = \langle \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n, v_i\rangle$

$= \alpha_i \langle v_i, v_i\rangle \qquad \because \langle v_i, v_j\rangle = 0 \; i \neq j$

$= \alpha_i \|v_i\|^2$

$= \alpha_i (1) \qquad \because \{v_1, v_2, \dots, v_n\}$ is orthonormal

$= \alpha_i$

Hence

$$U = \langle U, v_1\rangle v_1 + \langle U, v_2\rangle v_2 + \dots + \langle U, v_n\rangle v_n$$

Remarks# Any orthogonal system can be converted into an orthonormal system by normalizing the vectors.

Orthogonalization & Normalization Process#

Any linearly independent set of vectors in an inner-product space can be orthogonalized and hence can also be normalized because

because any orthogonal system can be converted into an orthonormal system

Theorem (Gram Schmidt orthogonalization Process)

Let $\{x_1, x_2, x_3, \ldots x_n, \ldots\}$ → ①

be any (countable) set of linearly independent (element) set in an inner product space V. Then V contains a set

$\{e_1, e_2, \ldots e_n \ldots\}$ → ②

Such that

(i) System ② is orthonormal

(ii) The spaces generated by $\{x_1, x_2 \ldots\}$ and by $\{e_1, e_2, \ldots\}$ are same

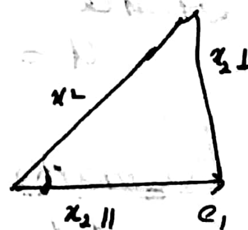
Proof# Step-I Consider x_1. We get e_1 by normalizing x_1 as

$$e_1 = \frac{x_1}{\|x_1\|} \quad \text{then } \|e_1\| = 1$$

Now any vector which is a L.C of x_1 is also a L.C of e_1 and Consequently the subspaces generated by x_1 & e_1 are same.

Step-II Now the vector

$x_2 - \langle x_2, e_1\rangle e_1$ is orthogonal to e_1.



Let $v_2 = x_2 - \langle x_2, e_1\rangle e_1$

and put

$$e_2 = \frac{v_2}{\|v_2\|} \quad \text{Then } \|e_2\| = 1$$

$$\langle e_2, e_1\rangle = \frac{1}{\|v_2\|}\{\langle x_2, e_1\rangle - \langle x_2, e_1\rangle\} = 0$$

Component of x_2 along e_1 be u. Then $u = \langle x_2, e_1\rangle \frac{e_1}{\|e_1\|} = \langle x_2, e_1\rangle$

Also spaces generated by $\{e_1, e_2\}$ & $\{x_1, x_2\}$ are same

Suppose we have constructed

$e_1, e_2, \ldots, e_{n-1}$ such that the space generated by $x_1, x_2, \ldots, x_{n-1}$ & $e_1, e_2, \ldots, e_{n-1}$ is same

nth – step

put

$$v_n = x_n - \langle x_n, e_1\rangle e_1 - \langle x_n, e_2\rangle e_2 - \langle x_n, e_3\rangle e_3 - \cdots - \langle x_n, e_{n-1}\rangle e_{n-1}$$

and let

$$e_n = \frac{v_n}{\|v_n\|} \quad \text{then } \|e_n\| = 1$$

So that $\{e_1, e_2, \ldots, e_n\}$ is an ortho-normal system. Also the space generated by $x_1, x_2, \ldots, x_n$ and $e_1, e_2, \ldots, e_n$ are the same. Hence the result.

Remarks # Since elements of a basis are L.I, therefore every non-zero finite dimensional inner product space has an orthonormal basis.

Example # Find the orthonormal system for linearly independent set.

$$\{1, \cos x, \cos 2x, \ldots, \cos nx, \ldots\}, \quad 0 \le x \le 2\pi$$

Sol # The inner product for the system is

$$\langle u_n, u_m\rangle = \langle \cos nx \cdot \cos mx\rangle$$

$$= \int_0^{2\pi} \cos nx \cos mx \, dx = \begin{cases} 0 & m \ne n \\ \pi & \text{if } m = n \\ 2\pi & \text{if } m = n = 0 \end{cases}$$

Here $e_1 = \frac{1}{\|1\|} = 1$

$$\|1\| = \left[\int_0^{2\pi} dx\right]^{1/2} = \frac{1}{\sqrt{2\pi}}$$

$$\therefore e_1 = \frac{1}{\sqrt{2\pi}}$$

Step – II $V_2 = x_2 - \langle x_2 \; e_1 \rangle e_1$

$= \cos x - \langle \cos x \; e_1 \rangle e_1$

$= \cos x - \langle \cos x \; \frac{1}{\sqrt{2\pi}} \rangle \cdot \frac{1}{\sqrt{2\pi}}$

$= \cos x - \frac{1}{\sqrt{2\pi}} \langle \cos x \cdot \frac{1}{\sqrt{2\pi}} \rangle$

$= (1 - \frac{1}{2\pi}) \cos x - 0$

Here

$\langle \cos x \; \frac{1}{\sqrt{2\pi}} \rangle = \int_0^{2\pi} \cos x \cdot \frac{1}{\sqrt{2\pi}} \, dx.$

$= \frac{1}{\sqrt{2\pi}} \left| \sin x \right|_0^{2\pi} = 0$

$V_2 = \cos x$

$e_2 = \frac{V_2}{\|V_2\|} = \frac{\cos x}{\|V_2\|}$

$\|V_2\|^2 = \langle \cos x \; \cos x \rangle$

$= \int_0^{2\pi} \cos^2 x \, dx = \int_0^{2\pi} \frac{1 + \cos 2x}{2} \, dx$

$= \left[\frac{1}{2} x + \frac{\sin 2x}{4} \right]_0^{2\pi}$

$= \pi$

$e_2 = \frac{\cos x}{\sqrt{\pi}}$

step – III

$V_3 = x_3 - \langle x_3 \; e_2 \rangle e_2 - \langle x_3 \; e_1 \rangle e_1$

$= \cos 2x - \langle \cos 2x \; \frac{\cos x}{\sqrt{\pi}} \rangle \frac{\cos x}{\sqrt{\pi}} - \langle \cos 2x \; \frac{1}{\sqrt{2\pi}} \rangle \cdot \frac{1}{\sqrt{2\pi}}$

$= \cos 2x$

$e_3 = \frac{V_3}{\|V_3\|} = \frac{\cos 2x}{\|V_3\|}$

$\|V_3\|^2 = \int_0^{2\pi} \cos^2 2x \, dx = \int_0^{2\pi} \left(\frac{1 + \cos 4x}{2} \right) dx$

$= \left[\frac{1}{2} x + \frac{\sin 4x}{8} \right]_0^{2\pi} = \pi$

$e_3 = \frac{\cos 2x}{\sqrt{\pi}}$ In general $e_n = \frac{\cos (n-1) x}{\sqrt{\pi}}$ $n > 1$

Example # Consider the space $\mathcal{L}^2[a, b]$, the space of all Lebesgue integrable functions defined on $[a, b]$. functions also square integrable i.e

$$\int_a^b |f(x)|^2 dx < \infty$$

For $f, g \in \mathcal{L}^2[a, b]$

$$\langle f, g \rangle = \int_a^b f \bar{g} \, dx \longrightarrow ①$$

It is easy to verify that ① defines an inner product in $\mathcal{L}^2$.

Consider the functions

$$f_n(x) = \sin nx \qquad x \in [-\pi, \pi], n = 1, 2 \ldots$$

Then the set $\{f_n\}$ is an orthogonal system.

Here

$$\langle f_n, f_m \rangle = \int_{-\pi}^{\pi} \sin nx \sin mx \, dx.$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} 2 \sin nx \sin mx \, dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} [\cos(n-m)x - \cos(n+m)x] \, dx.$$

$$= \frac{1}{2} \left[\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m} \right]_{-\pi}^{\pi}$$

$$= 0 \qquad \text{if } m \neq n$$

Similarly if $g_n(x) = \cos nx$ $\quad n = 0, 1 \ldots$, $\; x \in [-\pi, \pi]$

is also an orthogonal system.

Theorem # (Bessel's Inequality)

Let $\{e_1, e_2 \ldots e_n\}$ be a finite orthonormal set in an inner-product space V or Hilbert space. If x is any vector in V, then

$$\sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \leq \|x\|^2$$

Proof

$$0 \leq \| x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \|^2$$

$$= \langle x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i, \; x - \sum_{j=1}^{n} \langle x, e_j \rangle e_j \rangle$$

$$= \langle x, x \rangle - \langle x, \sum_{i=1}^{n} \langle x, e_i \rangle e_i \rangle - \langle \sum_{j=1}^{n} \langle x, e_j \rangle e_j, x \rangle$$

$$+ \langle \sum_{i=1}^{n} \langle x, e_i \rangle e_i, \sum_{j=1}^{n} \langle x, e_j \rangle e_j \rangle$$

$$= \|x\|^2 - \sum_{i=1}^{n} \overline{\langle x, e_i \rangle} \langle x, e_i \rangle - \sum_{j=1}^{n} \langle x, e_j \rangle \langle e_j, x \rangle$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \langle x, e_i \rangle \overline{\langle x, e_j \rangle} \langle e_i, e_j \rangle$$

$$= \|x\|^2 - \sum_{i=1}^{n} \overline{\langle x, e_i \rangle} \langle x, e_i \rangle - \sum_{j=1}^{n} \langle x, e_j \rangle \overline{\langle x, e_j \rangle}$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \langle x, e_i \rangle \overline{\langle x, e_j \rangle} \langle e_i, e_j \rangle$$

$$= \|x\|^2 - \sum_{i=1}^{n} |\langle x, e_i \rangle|^2 - \sum_{i=1}^{n} |\langle x, e_i \rangle|^2$$

$$+ \sum_{i=1}^{n} \langle x, e_i \rangle \overline{\langle x, e_i \rangle} \cdot 1 \qquad \text{for } j = i$$

$$= \|x\|^2 - 2\sum_{i=1}^{n} |\langle x, e_i \rangle|^2 + \sum_{i=1}^{n} |\langle x, e_i \rangle|^2$$

$$= \|x\|^2 - \sum_{i=1}^{n} |\langle x, e_i \rangle|^2$$

or $$\sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \leq \|x\|^2 \quad \text{(Proved)}$$

Remarks # The inner product $\langle x, e_i \rangle$ in Bessel inequality is known as Fourier Co-efficient w.r.t the orthonormal system $\{e_1, e_2, \ldots, e_n\}$

Theorem # If $\{e_i\}$ is an orthonormal set in an inner Product space V (or Hilbert space), then the set $S = \{e_i : \langle x, e_i\rangle \neq 0\}$ is either empty or countable.

Proof # For each +ve integer consider the set

$$S_n = \{e_i : |\langle x, e_i\rangle|^2 > \frac{\|x\|^2}{n}\}$$

By Bessel inequality, S_n contains at most $n-1$ vectors. The conclusion now follows from the fact

$$S = \bigcup_{n=1}^{\infty} S_n$$

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Theorem # (General Bessel's Inequality)

If $\{e_i\}$ is an orthonormal set in an inner product space V (or Hilbert space), then

$$\sum |\langle x, e_i\rangle|^2 \leq \|x\|^2 \quad \forall x \in V \rightarrow (1)$$

Proof # Let $S = \{e_i : \langle x, e_i\rangle \neq 0\}$. If S is empty we define $\sum |\langle x, e_i\rangle|^2$ to be number 0 and in this case (1) is clearly true.

Assume that S is non-empty. Then it must be finite or countably infinite.

Case-I If S is finite, it can be written in the form

$$S = \{e_1, e_2, \ldots, e_n\}$$

and in this case $\sum |\langle x, e_i\rangle|^2 = \sum_{i=1}^{n} |\langle x, e_i\rangle|^2$ which is clearly independent of the order in which the elements of S are arranged.

So

$$\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \leq \|x\|^2 \quad \text{as proved above}$$

in Bessel's inequality for finite no of orthonormal vectors

Case - II: Let S be countably infinite and the vectors in it be arranged in a definite order

$$S = \{e_1, e_2, \ldots, e_n, \ldots\}$$

By the theory of absolutely convergent series, if $\sum_{n=1}^{\infty} |\langle x, e_i\rangle|^2$ converges, then every series obtained from this by arranging its terms is also convergent and all such series have the same sum i.e $\sum |\langle x, e_i\rangle|^2$ is non-negative extended real number which depends only on S not on the arrangements of vectors. Hence we have.

$$\sum |\langle x, e_i\rangle|^2 = \sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2$$

Now by Bessel's inequality for finite orthonormal vectors, we have

$$\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \leq \|x\|^2 \quad \text{for each } n$$

i.e every partial sum of the series $\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2$ is such that it can not exceed $\|x\|^2$.

Therefore

$$\lim_{n\to\infty} \sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \leq \|x\|^2$$

$$\Rightarrow \sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 \leq \|x\|^2$$

(Proved)

Theorem# Let $\{e_1, e_2, \ldots, e_n\}$ be a finite orthonormal set in a Hilbert space H or inner product space H. If x is any vector in H then

$$x - \sum_{i=1}^{n} \langle x, e_i\rangle e_i \perp e_j \quad \forall j = 1, 2, \ldots n$$

Proof#

$$\langle x - \sum_{i=1}^{n} \langle x\ e_i \rangle e_i,\ e_j \rangle$$

$$= \langle x\ e_j \rangle - \sum_{i=1}^{n} \langle x\ e_i \rangle \langle e_i\ e_j \rangle$$

$$= \langle x\ e_j \rangle - \langle x\ e_j \rangle \qquad \text{for } i=j,\ \langle e_j\ e_j \rangle = 1$$

$$= 0 \qquad \text{for } i \neq j,\ \langle e_i\ e_j \rangle = 0$$

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Remarks # (1) The above says that if we substract from a vector its components in several perpendicular direction, then the result has no component left in any of these directions.

(2) Loose geometric interpretation of the Bessels Inequality is that the sum of squares of the components of a vector in various perpendicular directions does not exceede the square of the length of the vector itself.

Theorem # If $\{e_i\}$ is an orthonormal set in a Hilbert space, and if x is an arbitrary vector in H Then

$$x - \sum (x\ e_i) e_i \perp e_j \quad \text{for each } j$$

Proof# We define here $\sum (x\ e_i)e_i$ for various cases. let $S = \{e_i : (x\ e_i) \neq 0\}$

Case-I # When S is empty we define $\sum(x\ e_i)e_i$ to be the vector 0 and statement reduces to

$x - 0 = x$ is orthogonal to each e_j, which is precisely what is meant by saying that S is empty.

Case-II # When S is non-empty and finite and can

be written in the form

$$S = \{e_1, e_2 \ldots \ldots e_n\}$$

we define $\sum (x, e_i) e_i$ to be $\sum_{i=1}^{n} (x\ e_i) e_i$ and in this case statement reduces to above theorem

Case – III If S is countably infinite. Let the vectors in S be listed in a definite order

$$S = \{e_1, e_2, \ldots \ldots e_n, \ldots \ldots\}$$

we put $s_n = \sum_{i=1}^{n} (x\ e_i) e_i$ and for $m > n$

$$\|s_m - s_n\|^2 = \|\sum_{i=n+1}^{m} (x\ e_i) e_i\|^2 = \sum_{i=n+1}^{m} |\langle x\ e_i\rangle|^2$$

By Bessel's inequality $\sum_{i=n+1}^{m} |\langle x, e_i\rangle|^2$ converges. So $\{s_n\}$ is a cauchy sequence in H and since H is complete, this sequence converges to vector s i.e

$$s = \sum_{n=1}^{\infty} \langle x\ e_n\rangle e_n$$

We define $\sum \langle x, e_i\rangle e_i$ to be $\sum_{n=1}^{\infty} \langle x\ e_n\rangle e_n$. Then result follows from the continuity of inner product space

$$\langle x - \sum \langle x\ e_i\rangle, e_j\rangle = \langle x - s\ e_j\rangle = \langle x\ e_j\rangle - \langle s\ e_j\rangle$$

$$= \langle x\ e_j\rangle - \langle \lim_{n\to\infty} s_n\ e_j\rangle$$

$$= \langle x\ e_j\rangle - \lim_{n\to\infty} \langle s_n\ e_j\rangle = \langle x\ e_j\rangle - \langle x\ e_j\rangle = 0$$

Now we show that $\sum \langle x, e_i\rangle e_i$ is valid in the sense that it does not depend upon arrangement of vectors in S. Let the vectors in S be rearranged in any manner

$$S = \{f_1, f_2, \ldots \ldots f_n, \ldots \ldots\}$$

we put $s'_n = \sum_{i=1}^{n} (x\ f_i) f_i$ and we see as above the sequence $\{s'_n\}$ converges to a limit s' which we write in the form $s' = \sum_{n=1}^{\infty} (x, f_n) f_n$

we show that $s' = s$. Let $\epsilon > 0$ be given and n_0 be a +ve integer so large that if $n \geq n_0$, then

$$\| s_n - s \| < \epsilon \quad \& \quad \| s_n' - s' \| < \epsilon$$

and
$$\sum_{i=n_0+1}^{\infty} |\langle x, e_i \rangle|^2 < \epsilon^2$$

for some integer $m_0 > n_0$, all terms of s_{n_0} occur among those of s_{n_0}' so $s_{m_0}' - s_{n_0}$ is a finite sum of terms of the form $\langle x, e_i \rangle e_i$ for $i = n_0+1, n_0+2 \ldots$

This yields

$$\| s_{m_0}' - s_{n_0} \|^2 \leq \sum_{i=n_0+1}^{\infty} |\langle x, e_i \rangle|^2 < \epsilon^2$$

so $\| s_{m_0}' - s_{n_0} \| < \epsilon$

and

$$\| s' - s \| \leq \| s' - s_{m_0}' \| + \| s_{m_0}' - s_{n_0} \| + \| s_{n_0} - s \| < \epsilon + \epsilon + \epsilon = 3\epsilon$$

Since ϵ is arbitrary, this shows that $s' = s$.

Theorem # Let $\{e_1, e_2, e_3 \ldots e_n, \ldots\}$ be an orthonormal system in an inner product space V. Let x be an arbitrary element of V. Then for any scalars

$$a_1, a_2 \ldots \quad a_n \ldots$$

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The expression

$$\| x - \sum_{k=1}^{n} a_k e_k \|$$

assumes its min value for $a_k = c_k = \langle x, e_k \rangle$ $\quad k = 1, 2, \ldots n$

This minimum equals

$$\|x\|^2 - \sum_{k=1}^{n} |c_k|^2$$

Also

$$\sum_{k=1}^{\infty} |c_k|^2 \leq \|x\|^2 \quad (\text{Bessel's inequality})$$

Proof # Take $s_n = \sum_{k=1}^{n} a_k e_k$

Then using the orthonormality of $e_1, e_2, \ldots \ldots e_n$

∴ We have

$$0 \leq \|x - s_n\|^2 = \langle x - \sum_{i=1}^{n} a_i e_i \;\; x - \sum_{k=1}^{n} a_k e_k \rangle$$

$$= \|x\|^2 - \sum_{i=1}^{n} a_i \langle e_i \; x \rangle - \sum_{k=1}^{n} \bar{a}_k \langle x \; e_k \rangle$$

$$+ \sum_{k=1}^{n} \sum_{i=1}^{n} a_i \bar{a}_k \langle e_i \; e_k \rangle$$

$$= \|x\|^2 - \sum_{i=1}^{n} a_i \bar{c}_i - \sum_{k=1}^{n} \bar{a}_k c_k + \sum_{k=1}^{n} |a_k|^2$$

$$= \|x\|^2 - \sum_{k=1}^{n} |c_k|^2 + \sum_{k=1}^{n} |a_k - c_k|^2 \rightarrow (A)$$

The expression on R.H.S of (A) clearly attains its minimum value when

$$\sum_{k=1}^{n} |a_k - c_k|^2 = 0$$

i.e

when $a_k = c_k = \langle x \;\; e_k \rangle$

This minimum is given by

$$0 \leq \|x - s_n\| = \|x\|^2 - \sum_{k=1}^{n} |c_k|^2$$

so that

$$\sum_{k=1}^{n} |c_k|^2 \leq \|x\|^2$$

for any arbitrary natural number n. Hence the series

$$\sum_{k=1}^{\infty} |c_k|^2 \quad \text{is convergent}$$

and

$$\sum_{k=1}^{\infty} |c_k|^2 \leq \|x\|^2$$

Total Orthonormal Set

An orthonormal system $\{e_1, e_2 \ldots\ldots e_n \ldots\}$ of elements in an inner-product space V is said to be total (or closed) if for each $x \in V$ and $c_k = \langle x \; e_k \rangle$

$$\|x\|^2 = \sum_{k=1}^{\infty} |c_k|^2 \quad \text{(Parseval's equality)}$$

Theorem# An orthonormal system $\{e_1, e_2, \ldots, e_n, \ldots\}$ is an inner product space V is total iff for each $x \in V$,

$$x = \sum_{k=1}^{\infty} c_k e_k \qquad c_k = \langle x, e_k \rangle$$

Proof# The orthonormal system $\{e_1, e_2, \ldots e_n, \ldots\}$ is total iff

$$\|x\|^2 = \sum_{k=1}^{\infty} |c_k|^2 \Leftrightarrow \lim_{n \to \infty} \| x - \sum_{k=1}^{n} c_k e_k \| = 0$$

$$\Leftrightarrow x = \sum_{k=1}^{\infty} c_k e_k$$